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SOME PRIMAL PROBLEMS OF THE AXISYMMETRIC THEORY OF ELASTICITY FOR SPACE WITH A FLAT SLIT BOUNDED BY A CIRCLE

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We consider the first and second primal problems of the axisymmetric theory of elasticity for space with a round slit and a mixed problem in which forces are specified on one side of the slit and displacements are specified on the other side. The problems reduce to conjugation problems for generalized analytic functions on rectilinear segments, whose solution is obtained in closed form.

Reducing plane elastic problems for cracked bodies to conjugation problems for complex analytic functions at the crack edges is an effective method of solution. In some cases, spatial axisymmetric elastic problems with a flat boundary can be reduced to conjugation problems for analytic functions [1] or to conjugation problems for p-analytic functions [2]. Using generalized analytic functions (GAF) in problems of this type (space with a circular boundary of conditions, etc.) is also effective. Below, we give a solution of the primal elastic problems for space with a round slit in an axisymmetric case where forces or displacements are specified at the edges of the slit or forces are specified at one edge and displacements are specified at the other edge.

For the elastic characteristics we use the following designations: G is the shear modulus and ν is Poisson's ratio. The round slit of radius r is in the plane $z = 0$ (z, r, and θ are cylindrical coordinates). It is assumed that forces or displacements are specified at the edge of the slit. Using GAF, we write the boundary conditions at the edge of the slit [1]:

$$[\Phi'(\tau)]^{\pm} - [\overline{\Psi'(\tau)}]^{\pm} = \sigma_z^{\pm} + i\tau_{zr}^{\pm} = f^{\pm}(\tau) \qquad (\tau \in L_{\pm}) \tag{1}$$

or

$$æ[\Phi(\tau)]^{\pm} - [\overline{\Psi(\tau)}]^{\pm} = 2G(u_z^{\pm} + iu_r^{\pm}) = g^{\pm}(\tau) \qquad (\tau \in L_{\pm}). \tag{2}$$

Here $\Phi(t)$ and $\Psi(t)$ are generalized analytic functions over the entire plane, except on the segment L, $z = 0$, $0 < |r| < c$, σ_z and τ_{zr} are the normal and tangential stresses, respectively, u_z and u_r are the displacements along the z and r axes, $L_{\pm}$ denote the lower and upper sides of the slit, respectively, $æ = 3-4\nu$, and $t = z+ir$.

The differentiation operation for the GAF is defined by

$$\Phi'(t) = \lim_{t_1 \to t} \Big[\Phi(t_1) - \operatorname{Re}\Phi(t) - (ir/r_1)\operatorname{Im}\Phi(t)\Big]\Big/[z - z_1 + i(r - r_1)]. \tag{3}$$

In particular, $\Phi'(t) = \partial\Phi(t)/\partial z$. The stresses and displacements are given by the formulas

$$\sigma_z + \sigma_r + \sigma_\theta = 4(1+\nu)\operatorname{Re}\Phi'(t), \quad \sigma_\theta = 4\nu\operatorname{Re}\Phi'(t) + 2G\,u_r/r,$$
$$\sigma_z + i\tau_{zr} = \Phi'(t) - 2z\overline{\Phi''(t)} - \overline{\Psi'(t)}, \quad 2G(u_z + iu_r) = æ\Phi(t) - 2z\overline{\Phi'(t)} - \overline{\Psi(t)}. \tag{4}$$

Taking into account that $\overline{\Psi(\tau)} = \Psi(\bar{\tau})$ for $\tau = ir$, we write conditions (1) and (2) in the form

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$$[\Phi'(\tau)]^{\pm} - [\overline{\Psi'(\tau)}]^{\pm} = [\Phi'(\tau)]^{\pm} - [\Psi'(-\tau)]^{\pm} = f^{\pm}(\tau), \tag{5}$$

$$æ[\Phi(\tau)]^{\pm} - [\overline{\Psi(\tau)}]^{\pm} = æ\Phi^{\pm}(\tau) - \Psi^{\pm}(-\tau) = g^{\pm}(\tau). \tag{6}$$

Using the definition of the derivative (3), we differentiate both sides of equality (6) along the slit edge assuming that the derivatives exist up to the edge. As a result, we obtain $[æ\Phi^{\pm}(\tau) - \Psi^{\pm}(-\tau)]' = (g^{\pm}(\tau))' = g_1^{\pm}(\tau)$, where $g_1^{\pm}(\tau) = 2G(du_r^{\pm}/dr + u_r^{\pm}/r - i\, du_z^{\pm}/dr)$.

Denoting $\varphi(t) = \Phi'(t)$ and $\psi(t) = \Psi'(t)$, instead of (5) and (6) we have

$$\varphi^{\pm}(\tau) - \psi^{\pm}(-\tau) = f^{\pm}(\tau), \qquad æ\varphi^{\pm}(\tau) + \psi^{\pm}(-\tau) = g_1^{\pm}(\tau) \qquad (\tau \in L). \tag{7}$$

Let $t = z + ir$ and $\tau = ir$ be the interior and boundary points. The function $\psi_1(t) = \psi(-t)$ is also a GAF of the class considered here. For $t \to +0 + ir$, we have $-t \to -0 - ir$. Hence,

$$\psi^{-}(-\tau) = \psi^{-}(-ir) = \psi_1^{+}(ir) = \psi_1^{+}(\tau), \qquad \psi^{+}(-\tau) = \psi^{+}(-ir) = \psi_1^{-}(ir) = \psi_1^{-}(\tau). \tag{8}$$

Thus, conditions (7) are written as

$$\varphi^{\pm}(\tau) - \psi_1^{\mp}(\tau) = f^{\pm}(\tau), \qquad æ\varphi^{\mp}(\tau) + \psi_1^{\pm}(\tau) = g_1^{\mp}(\tau). \tag{9}$$

Adding together and subtracting the first two equalities of (9), we obtain

$$\begin{gathered}[\varphi(\tau) - \psi_1(\tau)]^{+} + [\varphi(\tau) - \psi_1(\tau)]^{-} = f^{+}(\tau) + f^{-}(\tau) = f_1(\tau), \\ [\varphi(\tau) + \psi_1(\tau)]^{+} - [\varphi(\tau) + \psi_1(\tau)]^{-} = f^{+}(\tau) - f^{-}(\tau) = f_2(\tau).\end{gathered} \tag{10}$$

In the plane with the hole intersecting the z axis, the regular GAF $\varphi(t)$ and $\psi_1(t)$ outside the hole and vanishing at infinity are written as [1]

$$\begin{gathered}\varphi(t) = S(\varphi_*(\zeta)) = -\frac{1}{\pi|r|}\int_{\bar{t}}^{t} \varphi_*(\zeta) M(\zeta, t)\, d\zeta, \\ \psi_1(t) = S(\psi_{1*}(\zeta)) = -\frac{1}{\pi|r|}\int_{\bar{t}}^{t} \psi_{1*}(\zeta) M(\zeta, t)\, d\zeta,\end{gathered} \tag{11}$$

where $M(\zeta, t) = \sqrt{(\zeta - \bar{t})/(\zeta - t)}$ and $\varphi_*(\zeta)$ and $\psi_{1*}(\zeta)$ are functions that are holomorphic in the region D and vanish at infinity, for which the following equality holds: $\lim\limits_{|\zeta|\to\infty} \zeta\varphi_*(\zeta) = \lim\limits_{|\zeta|\to\infty} \zeta\psi_{1*}(\zeta) = 0$. In relation (11), it is assumed that the line of integration is below the slit and the line of branching of the radical $M(\zeta, t)$.

For the indicated behavior of the functions $\varphi_*(\zeta)$ and $\psi_{1*}(\zeta)$ at infinity, the value of the integral in (11) does not depend on the method of integration. Therefore, for $t \to -0 + ir$ we can integrate over the upper side of the slit L_+:

$$\begin{aligned}[\varphi(\tau) - \psi_1(\tau)]^{+} &= -\frac{1}{\pi|r|}\int_{\bar{\tau}}^{\tau} (\varphi_*(\sigma) - \psi_{1*}(\sigma))^{+} M^{+}(\sigma, \tau)\, d\sigma \\ &= \frac{1}{\pi|r|}\int_{\bar{\tau}}^{\tau} (\varphi_*(\sigma) - \psi_{1*}(\sigma))^{+} M^{-}(\sigma, \tau)\, d\sigma.\end{aligned} \tag{12}$$

Here allowance is made for the equality $M^{+}(\sigma, \tau) = -M^{-}(\sigma, \tau)$. For the lower side of the slit, we have

$$[\varphi(\tau) - \psi_1(\tau)]^{-} = -\frac{1}{\pi|r|}\int_{\bar{\tau}}^{\tau} (\varphi_*(\sigma) - \psi_{1*}(\sigma))^{-} M^{-}(\sigma, \tau)\, d\sigma. \tag{13}$$

From (12) and (13), we obtain

$$[\varphi(\tau) - \psi_1(\tau)]^{+} + [\varphi(\tau) - \psi_1(\tau)]^{-} = \frac{1}{\pi|r|}\int_{\bar{\tau}}^{\tau} [\Omega^{+}(\sigma) - \Omega^{-}(\sigma)] M^{-}(\sigma, \tau)\, d\sigma, \tag{14}$$

where $\Omega^{\pm}(\sigma) = \varphi_*^{\pm}(\sigma) - \psi_{1*}^{\pm}(\sigma)$. Similarly, we obtain

$$[\varphi(\tau) + \psi_1(\tau)]^+ - [\varphi(\tau) + \psi_1(\tau)]^- = \frac{1}{\pi|r|} \int_{\bar{\tau}}^{\tau} [\Lambda^+(\sigma) + \Lambda^-(\sigma)] M^-(\sigma, \tau)\, d\sigma,$$
$$\Lambda^{\pm}(\sigma) = \varphi_*^{\pm}(\sigma) + \psi_{1*}^{\pm}(\sigma). \tag{15}$$

The operator S^{-1}, which is the inverse of S, has the form [1]

$$S^{-1}(\Phi(t)) = \frac{1}{2} \frac{d}{d\zeta} \int_{\bar{\zeta}}^{\zeta} \Phi(t) M(\zeta, t) h(t, \zeta)\, dt, \qquad h(t, \zeta) = \begin{cases} 1, & \operatorname{sign}(\operatorname{Im}\zeta \operatorname{Im} t) > 0, \\ -1, & \operatorname{sign}(\operatorname{Im}\zeta \operatorname{Im} t) < 0. \end{cases}$$

Applying the operator S^{-1} to both sides of equalities (14) and (15), we obtain the following conjugation problems for the analytic functions $\Omega(\zeta)$ and $\Lambda(\zeta)$ [3]:

$$\Omega^+(\zeta) - \Omega^-(\zeta) = -S^{-1}(f_1(\tau)) = F_1(\sigma), \quad \Lambda^+(\zeta) + \Lambda^-(\zeta) = -S^{-1}(f_2(\tau)) = F_2(\sigma). \tag{16}$$

For sufficiently large $|\zeta|$, the functions $\varphi_*(\zeta)$ and $\psi_{1*}(\zeta) = \psi_*(-\zeta)$ can be expanded in the series $\varphi_*(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^{-n}$ and $\psi_{1*}(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^{-n}$, where the coefficients obey the equalities

$$a_1 = b_1 = 0, \qquad \text{æ} a_2 - b_2 = 0. \tag{17}$$

Indeed, using the representations $\Phi(t) = S(\varphi_0(\zeta))$ and $\Psi(t) = S(\psi_0(\zeta))$, at infinity we obtain the expansions $\varphi_0(\zeta) = \sum_{n=1}^{\infty} a_n^0 \zeta^{-n}$, $\psi_0(\zeta) = \sum_{n=1}^{\infty} b_n^0 \zeta^{-n}$, where $\text{æ} a_1^0 + b_1^0 = 0$ [2]. Then, $\varphi(t) = \Phi'(t) = S(\varphi_0'(\zeta))$ and $\psi(t) = S(\psi_0(\zeta))$. On the symmetry axis [2],

$$\varphi(z) = \varphi_0'(z) \operatorname{sign}(z) = \operatorname{sign}(z)(-a_1^0/z^2 - 2a_2^0/z^3 - \ldots),$$
$$\psi(z) = \psi_0'(z) \operatorname{sign}(z) = \operatorname{sign}(z)(-b_1^0/z^2 - 2b_2^0/z^3 - \ldots).$$

Introducing $\psi_1(t) = \psi(-t)$, we have

$$\psi_1(z) = \operatorname{sign}(-z)(-b_1^0/z^2 - 2b_2^0/z^3 - \ldots) = \operatorname{sign}(z)(b_1^0/z^2 - 2b_2^0/z^3 + \ldots).$$

At the same time,

$$\varphi(t) = S(\varphi_*(\zeta)), \qquad \psi_1(t) = S(\psi_{1*}(\zeta)),$$
$$\varphi_*(\zeta) = a_2/\zeta^2 + a_3/\zeta^3 + \ldots, \qquad \psi_{1*}(\zeta) = b_2/\zeta^2 + b_3/\zeta^3 + \ldots,$$
$$\varphi(z) = \operatorname{sign}(z)(a_2/z^2 + a_3/z^3 + \ldots), \quad \psi_1(z) = \operatorname{sign}(z)(b_2/z^2 + b_3/z^3 + \ldots).$$

Comparing these expressions, we have $a_2 = -a_1^0$, $b_2 = b_1^0$, and, hence, $\text{æ} a_2 - b_2 = 0$.

The solution for the functions $\Omega(\zeta)$ and $\Lambda(\zeta)$ is written as

$$\Omega(\zeta) = \frac{1}{2\pi i} \int_L \frac{F_1(\sigma)}{\sigma - \zeta}\, d\sigma + \frac{c_1\zeta + c_0}{\zeta^2 + c^2} = L_1(F_1(\sigma)),$$
$$\Lambda(\zeta) = \frac{X(\sigma)}{2\pi i} \int_L \frac{F_2(\sigma)}{X^+(\sigma)(\sigma - \zeta)}\, d\sigma + X(\zeta) c_2 = L_2(F_2(\sigma)), \tag{18}$$

where $X(\zeta) = (\zeta^2 + c^2)^{-1/2}$, c_k are constants, and the second terms on the right sides of equalities (18) are solutions of the corresponding homogeneous conjugation problems (16). The coefficients c_k are easily found using conditions (17). Taking into account the formula for displacements (4) using the operator S, it is easy to verify that solutions of the form (18) satisfy the conditions of continuous displacements at the points $t = \pm ic$ and finite potential energy.

We write the specific form of the unknown coefficients ($k = 0$, 1, and 2). Since

$$\varphi_*(\zeta) = (\Omega(\zeta) + \Lambda(\zeta))/2, \qquad \psi_{1*}(\zeta) = (\Lambda(\zeta) - \Omega(\zeta))/2,$$

then, writing the integrals in (18) for $|\zeta| > c$ in the form

$$\int_L F_1(\sigma)/(\sigma - \zeta)\,d\sigma = -\zeta^{-1}\int_L F_1(\sigma)/(1 - \sigma\zeta^{-1})\,d\sigma = -\sum_{n=0}^{\infty}\Big(\int_L F_1(\sigma)\sigma^n\,d\sigma\Big)\zeta^{-n-1},$$

$$\int_L F_2(\sigma)/(X^+(\sigma)(\sigma - \zeta))\,d\sigma = -\zeta^{-1}\int_L F_2(\sigma)/(X^+(\sigma)(1 - \sigma\zeta^{-1}))\,d\sigma$$

$$= -\sum_{n=0}^{\infty}\Big(\int_L F_2(\sigma)\sigma^n/X^+(\sigma)\,d\sigma\Big)\zeta^{-n-1},$$

from equalities (18) we obtain the equations

$$a_1 = \lim_{\zeta\to\infty}\zeta[\Omega(\zeta) + \Lambda(\zeta)] = c_1 + c_2 - \frac{1}{2\pi i}\int_L F_1(\sigma)\,d\sigma = 0,$$

$$b_1 = \lim_{\zeta\to\infty}\zeta[\Lambda(\zeta) - \Omega(\zeta)] = c_2 - c_1 + \frac{1}{2\pi i}\int_L F_1(\sigma)\,d\sigma = 0,$$

$$æa_2 - b_2 = \lim_{\zeta\to\infty}\zeta^2[æ(\Omega(\zeta) + \Lambda(\zeta))/2 + (\Omega(\zeta) - \Lambda(\zeta))/2]$$

$$= -\frac{æ+1}{4\pi i}\int_L F_1(\sigma)\sigma\,d\sigma + \frac{æ+1}{2}c_0 - \frac{æ-1}{4\pi i}\int_L F_2(\sigma)/X^+(\sigma)\,d\sigma = 0.$$

Hence,

$$c_0 = \frac{1}{2\pi i}\Big(\int_L F_1(\sigma)\sigma\,d\sigma + \frac{æ-1}{æ+1}\int_L F_2(\sigma)/X^+(\sigma)\,d\sigma\Big), \quad c_1 = \frac{1}{2\pi i}\int_L F_1(\sigma)\,d\sigma, \quad c_2 = 0.$$

As an example, we consider the case where uniform pressure p is applied to the sides of the cut. In this case, $F_1(\sigma) = 2p$ and $F_2(\sigma) = 0$ and from formulas (18) it follows that $c_0 = c_2 = 0$, $c_1 = 2pc/\pi$, and, hence,

$$\Omega(\zeta) = \frac{p}{\pi i}\ln\frac{\zeta - ic}{\zeta + ic} + \frac{2pc}{\pi}\frac{\zeta}{\zeta^2 + c^2}, \qquad \Lambda(\zeta) = 0.$$

Consequently,

$$\varphi_*(\zeta) = -\psi_*(\zeta) = \frac{p}{2\pi i}\ln\frac{\zeta - ic}{\zeta + ic} + \frac{pc}{\pi}\frac{\zeta}{\zeta^2 + c^2} = \frac{p}{2\pi i}\Big(\ln\frac{\zeta - ic}{\zeta + ic} + \frac{ic}{\zeta + ic} + \frac{ic}{\zeta - ic}\Big).$$

Since $\lim_{|\zeta|\to\infty}\Phi_*(\zeta) = 0$ [1], for $\Phi_*(\zeta) = \int\varphi_*(\zeta)\,d\zeta$, we obtain

$$\Phi_*(\zeta) = \frac{p}{2\pi i}\Big(2ic + \zeta\ln\frac{\zeta - ic}{\zeta + ic}\Big).$$

Denoting $A(t) = \mathrm{Re}\,(\Phi(t))$, $B(t) = \mathrm{Im}\,(\Psi(t))$ and taking into account that $\varphi_*(\zeta) = -\psi_*(\zeta)$, we write formulas (4) as

$$\sigma_z + \sigma_r + \sigma_\theta = 4(1+\nu)A'(t), \quad \sigma_\theta = 4\nu A'(t) + 2(1-2\nu)B(t) + zB'(t)/r,$$

$$\sigma_z = 2A'(t) - 2zA''(t), \quad \tau_{rz} = 2z\frac{\partial^2 B(t)}{\partial z^2} \qquad (t = z + ir).$$

From formulas (11) we obtain

$$A'(t) = \mathrm{Re}\,(\Phi'(t)) = -\frac{1}{\pi|r|}\,\mathrm{Re}\Big(\int\limits_{\bar t}^{t} \Phi'_*(\zeta)\sqrt{\frac{\zeta-\bar t}{\zeta-t}}\,d\zeta\Big) = J_1(t) + J_2(t),$$

where

$$J_1(t) = -\frac{1}{\pi|r|}\,\mathrm{Re}\Big(\int\limits_{-r}^{r} \frac{p}{2\pi i}\ln\frac{\zeta+ic}{\zeta-ic}\,\frac{r}{\sqrt{r^2-y^2}}\,dy\Big),$$

$$J_2(t) = -\frac{1}{\pi|r|}\,\mathrm{Re}\Big(\int\limits_{-r}^{r} \frac{p}{2\pi i}\Big(\frac{ic}{\zeta+ic}+\frac{ic}{\zeta-ic}\Big)\frac{r}{\sqrt{r^2-y^2}}\,dy\Big)$$

$$(t = z+ir, \quad \zeta = x+iy, \quad x = z)$$

and integration is performed on the straight segment $(z-iy,\ z+iy)$. Using the residue theorem, for the corresponding branch of the radical, we obtain

$$J_2(t) = -\frac{cp}{\pi}\,\mathrm{Re}\,\frac{1}{\sqrt{r^2+(z+ic)^2}}.$$

To calculate the integral. $J_1(t)$, it is convenient to differentiate with respect to the parameter c and then use the residue theorem. Integrating over c, we obtain $J_1(t) = -(p/\pi)\,\mathrm{Im}\,[\ln(ic+z+\sqrt{r^2+(z+ic)^2}\,)]$. Thus,

$$A'(t) = -\frac{p}{\pi}\,\mathrm{Im}\,\Big[\ln(ic+z+\sqrt{r^2+(z+ic)^2}\,) + \frac{ic}{\sqrt{r^2+(z+ic)^2}}\Big].$$

Similarly, for $B(t)$ we obtain

$$B(t) = \mathrm{Im}\,(\Psi(t)) = -\frac{pr}{2\pi}\mathrm{Im}\,\Big[\ln(ic+z+\sqrt{r^2+(z+ic)^2}\,) + \frac{z-ic}{r^2}\sqrt{r^2+(z+ic)^2}\Big].$$

For $z = 0$ and $r > c$, we obtain the well-known solution [1]

$$\sigma_z = \frac{2p}{\pi}\Big(\frac{c}{\sqrt{r^2-c^2}} - \arcsin\frac{c}{r}\Big), \qquad \sigma_\theta = \frac{(1+2\nu)p}{\pi}\Big(\frac{c}{\sqrt{r^2-c^2}} - \arcsin\frac{c}{r}\Big) - \frac{(1-2\nu)pc^3}{\pi r^2\sqrt{r^2-c^2}},$$

$$\sigma_r = (1+2\nu)\sigma_z - \sigma_\theta = \frac{(1+2\nu)p}{\pi}\Big(\frac{c}{\sqrt{r^2-c^2}} - \arcsin\frac{c}{r}\Big) + \frac{(1-2\nu)pc^3}{\pi r^2\sqrt{r^2-c^2}}.$$

The conjugation conditions for the second boundary-value problem have the form

$$æ\varphi^{\pm}(\tau) + \psi_1^{\mp}(\tau) = g_1^{\pm}(\tau) \qquad (\tau \in L). \tag{19}$$

Adding together and subtracting equalities (19), we obtain

$$[æ\varphi(\tau)+\psi_1(\tau)]^+ + [æ\varphi(\tau)+\psi_1(\tau)]^- = g_0(\tau), \quad [æ\varphi(\tau)-\psi_1(\tau)]^+ - [æ\varphi(\tau)-\psi_1(\tau)]^- = g_2(\tau),$$

$$g_0(\tau) = g_1^+(\tau) + g_1^-(\tau), \qquad g_2(\tau) = g_1^+(\tau) - g_1^-(\tau) \qquad (\tau \in L).$$

As in the case of the first primal problem, for the functions $\Omega(\zeta) = æ\varphi_*(\zeta)+\psi_{1*}(\zeta)$ and $\Lambda(\zeta) = æ\varphi_*(\zeta)-\psi_{1*}(\zeta)$, we obtain the conjugation conditions

$$\Omega^+(\sigma) - \Omega^-(\sigma) = -S^{-1}(g_0(\tau)) = G_1(\sigma),$$
$$\Lambda^+(\sigma) + \Lambda^-(\sigma) = -S^{-1}(g_2(\tau)) = G_2(\sigma) \quad (\sigma \in L). \tag{20}$$

To satisfy conditions (17), as in the first boundary-value problem, it suffices to use the solutions $\Omega(\zeta)$ and $\Lambda(\zeta)$ in the form

$$\Omega(\zeta)=\frac{1}{2\pi i}\int\limits_{L}\frac{G_1(\sigma)\,d\sigma}{\sigma-\zeta}+\frac{c_1\zeta+c_0}{\zeta^2+c^2},$$

$$\Lambda(\zeta)=\frac{1}{2\pi iX(\zeta)}\int\limits_{L}\frac{X^+(\sigma)G_2(\sigma)\,d\sigma}{\sigma-\zeta}+\frac{c_2}{X(\zeta)},\quad X(\zeta)=((\zeta-ic)(\zeta+ic))^{1/2}.$$

The constants c_1 and c_2 are easily determined: $c_2=0$ and $c_1=\frac{1}{2\pi i}\int\limits_{L}G_1(\sigma)\,d\sigma$.

Since [1]

$$a_2=-P/(4\pi(1+æ)),\tag{21}$$

where P is the resultant of the forces applied to the slit, we can show that

$$c_0=-\frac{æP}{2\pi(1+æ)}+\frac{1}{2\pi i}\int\limits_{L}X^+(\sigma)G_2(\sigma)\,d\sigma+\frac{1}{2\pi i}\int\limits_{L}G_1(\sigma)\sigma\,d\sigma.$$

Therefore, for the resultant P, we can write the equality

$$P=2\pi\int\limits_{L}[(\sigma_z^++i\tau_{zr}^+)-(\sigma_z^-+i\tau_{zr}^-)]\tau\,d\tau$$

$$=-2\pi\int\limits_{L}[(\Phi'_+(\tau)-\Phi'_-(\tau))-(\overline{\Psi'_+(\tau)}-\overline{\Psi'_-(\tau)})]\tau\,d\tau$$

$$=-\frac{\pi}{æ}\int\limits_{L}S[(1-æ)(\Omega_+(\sigma)-\Omega_-(\sigma))-(1+æ)(\Lambda_+(\sigma)-\Lambda_-(\sigma))]\tau\,d\tau.$$

We consider another mixed boundary-value problem. Let displacements be specified on the lower side of the slit and forces be specified on the upper side of the slit. In this case, the conjugation conditions have the form

$$æ\varphi^-(\tau)+\psi_1^+(\tau)=g_1(\tau),\qquad \varphi^+(\tau)-\psi_1^-(\tau)=f(\tau)\qquad(\tau\in L),\tag{22}$$

where $g_1(\tau)$ and $f(\tau)$ are specified functions.

Multiplying the first of equalities of (22) by $-i/\sqrt{æ}$ and then by $i/\sqrt{æ}$ and adding, respectively, the right and left sides of the resulting equalities to the right and left sides of the second equality of (22), we have

$$\Big[\varphi(\tau)-\frac{i}{\sqrt{æ}}\psi_1(\tau)\Big]^+-i\sqrt{æ}\Big[\varphi(\tau)-\frac{i}{\sqrt{æ}}\psi_1(\tau)\Big]^-=f(\tau)-\frac{i}{\sqrt{æ}}g_1(\tau)=f_1(\tau),$$

$$\Big[\varphi(\tau)+\frac{i}{\sqrt{æ}}\psi_1(\tau)\Big]^++i\sqrt{æ}\Big[\varphi(\tau)+\frac{i}{\sqrt{æ}}\psi_1(\tau)\Big]^-=f(\tau)+\frac{i}{\sqrt{æ}}g_1(\tau)=f_2(\tau).$$

Performing calculations similar to the ones for the first and second boundary-value problems, for the analytic functions $\Omega(\zeta)=\varphi_*(\zeta)-(i/\sqrt{æ})\psi_{1*}(\zeta)$ and $\Lambda(\zeta)=\varphi_*(\zeta)+(i/\sqrt{æ})\psi_{1*}(\zeta)$, we obtain the conjugation problem

$$\begin{aligned}\Omega^+(\sigma)-i\sqrt{æ}\,\Omega^-(\sigma)&=-S^{-1}(f_1(\tau))=F_1(\tau),\\ \Lambda^+(\sigma)+i\sqrt{æ}\,\Lambda^-(\sigma)&=-S^{-1}(f_2(\tau))=F_2(\tau).\end{aligned}\tag{23}$$

From the equalities $\varphi_*(\zeta)=\overline{\varphi_*(\bar\zeta)}$ and $\psi_{1*}(\zeta)=\overline{\psi_{1*}(\bar\zeta)}$ it follows that

$$\Lambda(\zeta)=\overline{\Omega(\bar\zeta)}.\tag{24}$$

Therefore, the solutions of the conjugation problems (23) have the form $\Omega(\zeta)=L_1(F_1(\sigma))$ and $\Lambda(\zeta)=L_2(F_2(\sigma))$, where, with allowance for (24),

$$L_j(f(\sigma)) = \frac{X_j(\zeta)}{2\pi i} \int\limits_L \frac{f(\sigma)\,d\sigma}{X_j^+(\sigma)(\sigma-\zeta)} + X_j(\zeta)P_j(\zeta),$$

$$X_1(\zeta) = \overline{X_2(\bar{\zeta})} = (\zeta + ic)^{-\gamma}(\zeta - ic)^{\gamma-1} \qquad (j = 1, 2); \tag{25}$$

$$P_1(\zeta) = (c_1\zeta + c_0)/(\zeta - ic), \qquad P_2(\zeta) = (c_2\zeta + c_3)/(\zeta + ic),$$

$$\gamma = 3/4 + \ln æ/(4\pi i), \qquad c_1 = \bar{c}_2 = 0, \qquad c_0 = \bar{c}_3, \tag{26}$$

$$\mathrm{Re}\,[(æ - i\sqrt{æ})c_0] = \mathrm{Re}\,[(æ + i\sqrt{æ})c_3] = \mathrm{Re}\Big[(æ + i\sqrt{æ})/(2\pi i) \int\limits_L F_1(\sigma)/X_1^+(\sigma)\,d\sigma\Big].$$

Denoting the resultants of the forces on both sides of the slit and on the upper side by P and P_1, respectively, and taking into account (21) and (26), we obtain

$$\mathrm{Re}\,c_0 = \mathrm{Re}\Big[\frac{1}{2\pi i}\int\limits_L F_1(\sigma)/X_1^+(\sigma)\,d\sigma - \frac{P}{4\pi(1+æ)}\Big]. \tag{27}$$

Indeed,

$$a_2 = -\frac{P}{4\pi(1+æ)} = \frac{1}{2}\lim_{|\zeta|\to\infty} \zeta^2[\Omega(\zeta) + \Lambda(\zeta)]$$

$$= \frac{1}{2}\Big[-\frac{1}{2\pi i}\int\limits_L F_1(\sigma)/X_1^+(\sigma)\,d\sigma - \frac{1}{2\pi i}\int\limits_L F_2(\sigma)/X_2^+(\sigma)\,d\sigma + c_0 + c_3\Big];$$

whence formula (27) follows.

For the resultant P, we write the equality

$$P = 2\pi\int\limits_L (\sigma_z^+ - \sigma_z^-)r\,dr = 2\pi\int\limits_L \sigma_z^+ r\,dr - 2\pi\int\limits_L \mathrm{Re}\,(\Phi'_- - \Psi'_-)r\,dr$$

$$= P_1 - 2\pi\int\limits_L \mathrm{Re}\,[S((\varphi_*(\sigma))_- - (\psi_{1*}(\sigma))_+)]r\,dr$$

$$= P_1 - \pi\int\limits_L \mathrm{Re}\Big[S\Big(\Omega_1^-(\sigma) + \overline{\Omega_1^-(\bar{\sigma})} + X_1^-(\sigma)\,\frac{c_0}{\sigma - ic} + \overline{X_1^-(\bar{\sigma})}\,\frac{\bar{c}_0}{\sigma + ic}\Big)$$

$$-i\sqrt{æ}\Big(\Omega_1^+(\sigma) - \overline{\Omega_1^+(\bar{\sigma})} + X_1^+(\sigma)\,\frac{c_0}{\sigma - ic} - \overline{X_1^+(\bar{\sigma})}\,\frac{\bar{c}_0}{\sigma + ic}\Big)\Big]r\,dr, \tag{28}$$

where

$$\Omega_1(\zeta) = \frac{X_1(\zeta)}{2\pi i}\int\limits_L \frac{F_1(\sigma)\,d\sigma}{X_1^+(\sigma)(\sigma - \zeta)}.$$

This formula, together with (26) and (27), allows us to find c_0 and P. Using a solution in the form (25), it is easy to verify that the displacements are continuous and the potential energy is finite.

To calculate $\Phi(\zeta)$ and $\Psi(\zeta)$, we use the integration formula for GAF [1]:

$$\Phi(t) = \frac{1}{2}\int\limits_{t_0}^{t}\Big[\Big(1 + \frac{\tau - \bar{\tau}}{t - \bar{t}}\Big)\varphi(\tau)\,d\tau + \Big(1 - \frac{\tau - \bar{\tau}}{t - \bar{t}}\Big)\overline{\varphi(\tau)}\,d\bar{\tau}\Big] + A_1 + \frac{B_1}{r},$$

$$\Psi(t) = \frac{1}{2}\int\limits_{t_0}^{t}\Big[\Big(1 + \frac{\tau - \bar{\tau}}{t - \bar{t}}\Big)\psi_1(-\tau)\,d\tau + \Big(1 - \frac{\tau - \bar{\tau}}{t - \bar{t}}\Big)\overline{\psi_1(-\tau)}\,d\bar{\tau}\Big] + A_2 + \frac{B_2}{r}.$$

Here integration is performed along a curve that connects a certain point t_0 with the point t ($\operatorname{Im} t > 0$) and does not intersect L, and A_j and B_j ($j = 1, 2$) are real constants for which the following relation [1] holds:

$$æB_1 + B_2 = 0.$$

One of the constants A_j ($j = 1, 2$) can be specified arbitrarily, and the second is determined from the known displacements at any point. Without loss of generality of the solution, we can set $B_1 = B_2 = 0$. For $\operatorname{Im} t < 0$, the functions $\Phi(t)$ and $\Psi(t)$ can be determined from the formulas $\Phi(t) = \overline{\Phi(\bar{t})}$ and $\Psi(t) = \overline{\Psi(\bar{t})}$. Thus, all the required functions are obtained.

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